Flying Matrices

Intent

Students begin to develop the definition of multiplication of matrices.

Mathematics

The definition of matrix addition is fairly straightforward, while matrix multiplication is more complicated and somewhat harder to motivate. The activity *Flying Matrices* motivates the definition of matrix multiplication by having students examine the arithmetic involved through the context of a familiar problem.

Progression

Through this activity and *Matrices in the Oven*, students will establish the formal definition of matrix multiplication.

Approximate Time

25 to 30 minutes for activity

15 minutes for discussion

Classroom Organization

Small groups, followed by whole-class discussion

Doing the Activity

No introductory discussion is needed for this activity. Students will likely come up with nonstandard ways to set up these problems. That's okay— they'll learn the standard procedure in the discussion of the next activity, *Matrices in the Oven*. They should see that the standard procedure gives the same results as their alternate methods.

Bring the class together for discussion when all groups have at least finished Questions 1 through 4, and some are finished with the entire activity.

Discussing and Debriefing the Activity

For each of Questions 1 through 3 of *Flying Matrices*, have two or three students report on how they set up the matrices. As you get various answers, record them on overhead transparencies or chart paper so you can refer to them later.

Be prepared for a wide variety of methods for setting up matrices, perhaps including methods that are internally inconsistent. Although this variety may create some confusion, it will help students see that some steps they take in the setup process are arbitrary, while others need to be done a certain way. As students gain experience, they will see what makes sense.

Here is one *standard* way to set up Questions 1 through 3. (Labels have been added for clarity.)

For Question 1:

$$\begin{array}{r} \\ \text{feed} \\ \text{calcs} \end{array}\!\!\begin{array}{c} \begin{array}{cc} \text{wt} & \text{vol} \end{array} \\ \begin{bmatrix} 40 & 2 \\ 50 & 3 \end{bmatrix} \end{array}$$

For Question 2:

$$\begin{array}{r} \\ \text{Mon} \end{array}\!\!\begin{array}{c} \begin{array}{cc} \text{feed} & \text{calcs} \end{array} \\ \begin{bmatrix} 500 & 200 \end{bmatrix} \end{array}$$

For Question 3:

$$\begin{array}{r} \\ \text{Mon} \end{array}\!\!\begin{array}{c} \begin{array}{cc} \text{wt} & \text{vol} \end{array} \\ \begin{bmatrix} 30{,}000 & 1600 \end{bmatrix} \end{array}$$

But students certainly could have arranged the numbers differently. For example, Question 1 could be set up this way.

$$\begin{array}{r} \\ \text{wt} \\ \text{vol} \end{array}\!\!\begin{array}{c} \begin{array}{cc} \text{feed} & \text{calcs} \end{array} \\ \begin{bmatrix} 40 & 50 \\ 2 & 3 \end{bmatrix} \end{array}$$

Also, at this point, the only reason the *feed* column (or row) comes before the *calculator* column (or row) is that the information was presented in that order in the problem. As long as the four numbers are included in an organized fashion, the answer should be considered correct at this point. Similarly, the answers to Questions 2 and 3 could be presented as column vectors instead of row vectors.

Presumably, students will all come up with the same numerical totals for Question 3, even if they display the information differently.

Question 4

The explanations in Question 4 are a crucial part of this activity, so have several students describe what they did.

No matter how students displayed the information from Questions 1 through 3, they should have multiplied the appropriate entries of the matrix for Question 2 by the appropriate entries of the matrix for Question 1 and added the products. For example, they should see that the entry "30,000" in the matrix for Question 3 comes from the computation

$$(500 \cdot 40) + (200 \cdot 50)$$

Before proceeding further with the analysis of this computation, identify the matrix answer to Question 2 as the special type of matrix called a *vector*. **What do you call a matrix with either only one row or only one column?** This is either a row vector or a column vector, depending on how each student expressed the answer. We expressed this answer as the **row vector** $\begin{bmatrix} 500 & 200 \end{bmatrix}$.

Similarly, the two entries comprising the weight information also form either a row or column vector. According to the form we used for Question 2, this information gives the **column vector** $\begin{bmatrix} 40 \\ 50 \end{bmatrix}$.

Next ask students, **What was done with the two vectors $\begin{bmatrix} 500 & 200 \end{bmatrix}$ and $\begin{bmatrix} 40 \\ 50 \end{bmatrix}$ to produce the entry '30,000' in Question 3?** Help students to articulate an answer something like this:

You multiply the first entry of one by the first entry of the other, then multiply the second entry of one by the second entry of the other, and then add the two products.

Now move on to Questions 5 and 6. Again, there are likely to be a variety of answers. But you may want to suggest that the answer to Question 5 should be similar to that for Question 2. For example, if a student gave the answer to Question 2 as a 1 x 2 row vector, then the answer to Question 5 should contain two rows, one for each day, with the first row like the answer from Question 2. The answers might look like this:

For Question 5:

$$\begin{array}{r} \\ \text{Mon} \\ \text{Tues} \end{array}\begin{array}{c} \quad\text{feed}\quad\text{calcs} \\ \begin{bmatrix} 500 & 200 \\ 400 & 300 \end{bmatrix} \end{array}$$

For Question 6:

$$\begin{array}{r} \\ \text{Mon} \\ \text{Tues} \end{array}\begin{array}{c} \quad\text{wt}\qquad\text{vol} \\ \begin{bmatrix} 30{,}000 & 1600 \\ 31{,}000 & 1700 \end{bmatrix} \end{array}$$

Again, the key element of the discussion is explaining how the matrix answer to Question 6 comes from the matrix answers for Questions 1 and 5. For example, the entry "1700" represents the volume transported on Tuesday and comes from the computation

$$(400 \cdot 2) + (300 \cdot 3)$$

Using our notation, this can be seen as the product of the *Tuesday row vector* $\begin{bmatrix} 400 & 300 \end{bmatrix}$ (from the matrix of Question 5) and the *volume column vector* $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (from the matrix of Question 1).

Be sure students know what each of the numbers in the answer to Question 6 represents. For example, they should be able to express that the entry "31,000" represents the weight of Linda Sue's load on Tuesday.

Tell students at this point that the matrix that results from combining the answers for Questions 1 and 5 in this way is called the *product* of the matrices. Let them know that they will be seeing a standard way to set up matrices so that this multiplication can be done routinely on calculators or computers.

Key Questions

What do you call a matrix with either only one row or only one column?

What was done with the two vectors $\begin{bmatrix} 500 & 200 \end{bmatrix}$ and $\begin{bmatrix} 40 \\ 50 \end{bmatrix}$ to produce the entry '30,000' in Question 3?

Matrices in the Oven

Intent

Students continue to develop a definition of matrix multiplication.

Mathematics

Matrices in the Oven returns to the Woos' bakery (from *More Cookies*) to help students develop the logic behind matrix multiplication. The discussion leads to the realization that while there are many ways to define matrix multiplication that might make sense, a standard convention is needed in order for us to be able to understand one another's work.

Progression

This activity is similar to *Flying Matrices*. The discussion following the activity formally defines matrix multiplication. Students should record their results from this activity for use in *Fresh Ingredients*.

Approximate Time

30 minutes for activity (at home or in class)

40 to 50 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Students do this activity independently, with no introduction. The discussion will establish the procedure for multiplying matrices

Discussing and Debriefing the Activity

Give groups some time to discuss the activity, and then have students present each problem. Again, save the answers for use later.

Here is a possible set of answers for Questions 1 through 3:

For Question 1:

$$\begin{array}{r} \\ \\ \text{dough} \\ \text{icing} \\ \text{choc chip} \end{array} \begin{array}{c} \begin{array}{ccc} \text{plain} & \text{iced} & \begin{array}{c}\text{choc}\\ \text{chip}\end{array} \end{array} \\ \begin{bmatrix} 1 & 0.7 & 0.9 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.15 \end{bmatrix} \end{array}$$

For Question 2:

$$\begin{array}{r} \\ \\ \text{Wed} \\ \text{Thurs} \end{array} \begin{array}{c} \begin{array}{ccc} \text{plain} & \text{iced} & \begin{array}{c}\text{choc}\\ \text{chip}\end{array} \end{array} \\ \begin{bmatrix} 30 & 45 & 30 \\ 28 & 32 & 25 \end{bmatrix} \end{array}$$

For Question 3:

$$\begin{array}{r} \\ \\ \text{Wed} \\ \text{Thurs} \end{array} \begin{array}{c} \begin{array}{ccc} \text{dough} & \text{icing} & \begin{array}{c}\text{choc}\\ \text{chip}\end{array} \end{array} \\ \begin{bmatrix} 88.5 & 18 & 4.5 \\ 72.9 & 12.8 & 3.75 \end{bmatrix} \end{array}$$

Again, emphasize that the information in Questions 1 and 2 could be set up differently but that the computations that create the entries for the Question 3 matrix are the same no matter how the matrices are set up. For example, the entry "88.5" in the Question 3 matrix represents the amount of cookie dough used on Wednesday and comes from the computation $(30 \cdot 1) + (45 \cdot 0.7) + (30 \cdot 0.9)$.

Using the set-up given above for the matrices in Questions 1 and 2, students should see that they have taken the *Wednesday row* $\begin{bmatrix} 30 & 45 & 30 \end{bmatrix}$ from the matrix for Question 2 and multiplied its entries by the corresponding entries of the *cookie dough row* $\begin{bmatrix} 1 & 0.7 & 0.9 \end{bmatrix}$ from the matrix for Question 1. This computation is the same no matter how the matrices are set up. All that changes is the arrangement of the data in the matrix.

Be sure that students can identify what the numbers in their matrices represent, especially for Question 3.

Coordinating the Matrix Labels

Students should see that in both *Flying Matrices* and *Matrices in the Oven*, they found each entry of the final matrix by multiplying the entries of one row or column by the corresponding entries of another row or column and then adding the products. In other words, they multiplied two vectors.

At this point, tell students that mathematicians have agreed on a system for setting up matrices so that they can take unlabeled matrices and know how to combine them to get the product. The only question is whether to combine a row with a row, a row with a column, or a column with a column. Students should see that depending on how they set up their matrices, different combinations are appropriate.

To give some sense to this system, ask, **What categories of labels were used in the activity?** Help students to see that the labels are of three types:

- Type of cookie
- Type of ingredient
- Day of the week

Point out that for each matrix, they must decide on a category of label to assign to either row or column.

You might comment that it would be nice if we could assign either *row* or *column* to each of these throughout the problem. Point out that this is impossible, because different combinations occur at different stages of the problem.

For example, the category *type of cookie* occurs in the answers to both Question 1 and Question 2. Therefore, if this were set up as a row category in Questions 1 and 2, then *day of the week* and *type of ingredient* would have to be column categories. But these two occur together in Question 3, so we have an impasse.

Ask, **What relationship is there among the categories when two matrices are multiplied?** Guide students to articulate that whenever two matrices are multiplied, they have a *common category*. In *Matrices in the Oven*, the common category in Questions 1 and 2 is *type of cookie*. Point out that this category "drops out" in the final result.

Optional Approach

There is an informal way to see that it makes sense for *type of cookie* to be a row category in one of these two problems and a column category in the other. The reasoning is this:

Question 1 asks for *amount of a type of ingredient per type of cookie*. In a sense, the "units" for this matrix are like $\frac{\text{ingredients}}{\text{cookie type}}$, with *cookie type* in the denominator. But Question 2 asks for *amount of a type of cookie per day of the week*, so the units are $\frac{\text{cookie type}}{\text{day of the week}}$, with *cookie type* in the numerator.

Because *cookie type* is a numerator in one case and a denominator in the other, it makes some sense that *cookie type* should be a row category in one case and a column category in the other. When the matrices are multiplied, the "units" are also multiplied, with *cookie type* canceling out, resulting in $\frac{\text{ingredients}}{\text{day of the week}}$, which gives the units for Question 3. Thus, the computation can be represented in terms of units as

$$\frac{\text{ingredients}}{\text{cookie type}} \cdot \frac{\text{cookie type}}{\text{day of the week}} = \frac{\text{ingredients}}{\text{day of the week}}$$

Though this is not a formal mathematical explanation, it does have a mathematical basis, and it may be helpful in making sense of the formal definition of matrix multiplication.

Ask students what the categories were in *Flying Matrices*. They might refer to them as *day of the week*, *type of load*, and *type of plane limitation* (that is, weight or volume). They should also see that the common category was *type of load*. Therefore, again, the final answer simply involves the other two categories, namely, amount of weight or volume carried on each day.

The Definition of Matrix Multiplication

Tell students that for the sake of standardization, it is useful to have a rule for multiplying matrices so they can work with the numbers without thinking about the labels. The trick is to set up the matrices properly so that the arithmetic does what they want.

Give students this rule for the actual multiplication:

> **To get each individual entry of the product of two matrices, multiply a row in the first matrix by a column in the second matrix.**

Point out that as always when multiplying vectors, certain dimensions must match up. Here, in order for this computation to make sense, the *length* of a row in the first matrix must be the same as the *height* of a column in the second matrix.

Illustrate how this works in context using the *Matrices in the Oven* problem. You can begin by going back to an individual entry of the product in which students found the *cookie dough on Wednesday* entry using this computation:

$$(30 \cdot 1) + (45 \cdot 0.7) + (30 \cdot 0.9)$$

For this computation to match the definition, we need to make the information "30, 45, 30" (the amount of each cookie type for Wednesday) into a row of the first matrix, and the information "1, 0.7, 0.9" (the amount of cookie dough needed for each cookie type) into a column of the second matrix.

Ask students how to set up the matrices to fit this requirement. After a few minutes' work, they should see that the matrices can be set up like this:

$$\begin{array}{r} \\ \text{Wed} \\ \text{Thurs} \end{array}\begin{array}{c} \begin{array}{ccc} \text{plain} & \text{iced} & \text{choc chip} \end{array} \\ \begin{bmatrix} 30 & 45 & 30 \\ 28 & 32 & 25 \end{bmatrix} \end{array} \times \begin{array}{r} \\ \text{plain} \\ \text{iced} \\ \text{choc chip} \end{array}\begin{array}{c} \begin{array}{ccc} \text{dough} & \text{icing} & \text{choc chip} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0.7 & 0.4 & 0 \\ 0.9 & 0 & 0.15 \end{bmatrix} \end{array}$$

It turns out that the labels for the common category, *type of cookie*, go across in the first matrix and down in the second. Each row of the first matrix has three entries, because there are three types of cookies. For the same reason, each column of the second matrix has three entries.

The product of the two matrices has the row labels of the first matrix as its row labels, and the column labels of the second matrix as its column labels:

$$\begin{array}{r} \\ \text{Wed} \\ \text{Thurs} \end{array}\begin{array}{c} \begin{array}{ccc} \text{dough} & \text{icing} & \text{choc chip} \end{array} \\ \begin{bmatrix} 88.5 & 18 & 4.5 \\ 72.9 & 12.8 & 3.75 \end{bmatrix} \end{array}$$

With these matrices to refer to, illustrate again how, when we find the product matrix, we take a row from the first matrix, multiply its entries by the corresponding entries from a column in the second matrix, and add the products.

Point out that all students are doing is finding what was previously called the product of two vectors.

In this case, they are multiplying a row vector from the first matrix by a column vector from the second matrix. Tell them that this is referred to as *multiplying the row times the column.*

They should see that the position of the row and column being used tells us where to put the result in the final matrix. Thus, if we multiply the second row of the first matrix by the third column of the second matrix, the result goes in the second row, third column of the resulting matrix.

In our example, this means that when we multiply the *Thursday row* $\begin{bmatrix} 28 & 32 & 25 \end{bmatrix}$ by the *chocolate chip column* $\begin{bmatrix} 0 \\ 0 \\ 0.15 \end{bmatrix}$, we compute

$$(28 \cdot 0) + (32 \cdot 0) + (25 \cdot 0.15) = 3.75$$

This becomes the *Thursday, chocolate chip* entry of the result, which goes in the second row of the third column of the product matrix.

Ask, **What does the number 3.75 represent?** Students should see that it tells us that the Woos used 3.75 pounds of chocolate chips on Thursday.

Turning the Computation Around

Note that the original matrices could have been written with the rows and columns interchanged, resulting in this computation:

$$\begin{array}{l} \\ \text{dough} \\ \text{icing} \\ \text{choc chip} \end{array} \begin{array}{c} \begin{array}{ccc} \text{plain} & \text{iced} & \text{choc chip} \end{array} \\ \begin{bmatrix} 1 & 0.7 & 0.9 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.15 \end{bmatrix} \end{array} \times \begin{array}{l} \\ \text{plain} \\ \text{iced} \\ \text{choc chip} \end{array} \begin{array}{c} \begin{array}{cc} \text{Wed} & \text{Thurs} \end{array} \\ \begin{bmatrix} 30 & 28 \\ 45 & 32 \\ 30 & 25 \end{bmatrix} \end{array} = \begin{array}{l} \\ \text{dough} \\ \text{icing} \\ \text{choc chip} \end{array} \begin{array}{c} \begin{array}{cc} \text{Wed} & \text{Thurs} \end{array} \\ \begin{bmatrix} 88.5 & 72.9 \\ 18 & 12.8 \\ 4.5 & 3.75 \end{bmatrix} \end{array}$$

There is no theoretical reason to prefer one of these representations to the other.

Practice As Time Allows

If you have time available, give students some matrices to multiply. (Be sure to give them matrices whose dimensions match appropriately for multiplication.)

Key Questions

What categories of labels were used in the activity?

What relationship is there among the categories when two matrices are multiplied?

What does the number 3.75 represent?