Profitable Pictures

Intent

This activity continues students' work from *Picturing Pictures,* explicitly addressing the maximizing of profit.

Mathematics

In a linear programming problem with two variables, the set of constraints may be written as a system of linear inequalities. In this problem, that system is

$$5p + 15w \leq 180$$
$$p + w \leq 16$$
$$p \geq 0$$
$$w \geq 0$$

The graphical solution to this system defines a **feasible region**, a collection of points that form a polygon in the plane. The profit is also a linear function of p and w.

$$\text{Profit} = 40p + 100w$$

The central ideas of this activity are (1) that the set of points that produce a particular profit lie on the same line and (2) that the maximum profit is found by sliding this **profit line** up (increasing the profit) until it reaches the edge of the feasible region.

Progression

Students work on the activity and compare results in groups and as part of an extensive class discussion.

Approximate Time

55 minutes

Classroom Organization

Groups, followed by whole-class discussion

Materials

Profitable Pictures blackline master

Doing the Activity

Have students work in groups on the activity. Remind them that although they will work in groups, they will be preparing individual written reports.

If groups have trouble finding combinations that yield a specific profit, you might suggest that they consider points both outside and inside the feasible region.

For Question 5, you may have to ask questions to help groups formulate an explanation, such as, **What do you notice about combinations that produce a given profit? What happens as the profit increases?**

When most groups have had some time to explore Question 5, begin the discussion.

Discussing and Debriefing the Activity

Start the discussion by projecting the transparency of the graph from *Picturing Pictures* and having volunteers each mark a point from Question 2 to show a way in which Hassan can earn exactly $1,000. The only whole-number points in the feasible region that give this profit are (0, 10), (5, 8), and (10, 6).

Then have various students mark their points for Question 3, using a different color. The whole-number points in the feasible region that give a profit of exactly $500 are (0, 5), (5, 3), and (10, 1).

Finally, turn to Question 4. There are four whole-number points in the feasible region that yield a profit of $600: (0, 6), (5, 4), (10, 2), and (15, 0).

The Points for Each Profit Are Collinear

After all three sets of points have been plotted, ask the class, **What do you notice about the different sets of points?** Students should recognize that the points for each amount of profit lie on a straight line.

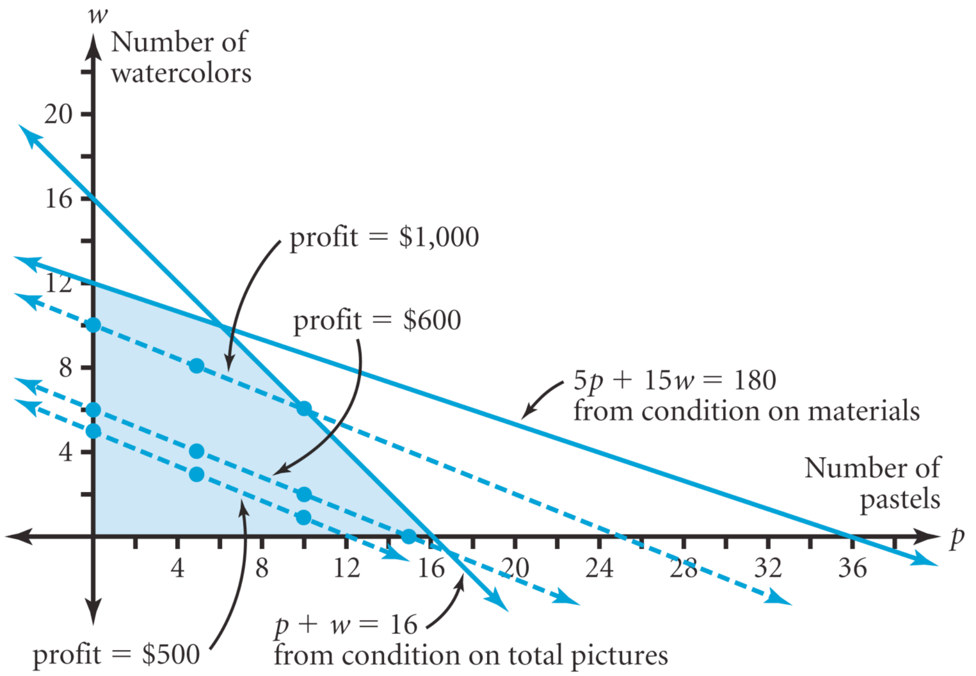
Introduce the term **profit line** for the set of points with a given profit. Have students draw in the complete lines on their graphs, connecting the individual points they found in Questions 2 through 4 and labeling each line with the profit it represents. On the transparency, you can extend the profit lines to include points outside the feasible region. Note that the profit lines include points whose coordinates are not whole numbers, even though Hassan can't make fractions of pictures.

Ask students, **Why do the points for a given profit lie on a straight line?** If necessary, ask what condition the coordinates must satisfy for a point to give a profit, for example, of $1,000.

Previously, students identified the profit expression as $40p + 100w$. Now they should realize that for a combination of pictures to yield a profit of $1,000, the number pair must satisfy the equation $40p + 100w = 1000$. In other words, the set of number pairs that give a profit of $1,000 is the same as the set of solutions to the equation $40p + 100w = 1000$, and the points corresponding to these pairs is the graph of the equation.

The diagram should now look something like the following.

Profit Lines for
Profitable Pictures



Profit Lines Are Parallel

A crucial element of this analysis is that the profit lines are all parallel. If it hasn't yet been mentioned, ask, **What do you notice about the set of profit lines?** Students should recognize that these are parallel lines and that as the profit increases the line "slides" upward to the right.

What does it mean for lines to be parallel? Bring out the key idea that parallel lines have no points in common.

Ask students how the algebraic representations of the three profit lines reinforce the visual evidence that the lines are parallel.

$40p + 100w = 1000$ (for a profit of $1,000)

$40p + 100w = 500$ (for a profit of $500)

$40p + 100w = 600$ (for a profit of $600)

Students may reason about this in various ways. One approach is to observe that the equations $40p + 100w = 1000$ and $40p + 100w = 500$ cannot have any solutions in common; for any values of p and w, the expression $40p + 100w$ cannot equal both 1000 and 500. Because these linear equations have no common solutions, the graphs have no common points, which means their lines must be parallel.

Maximizing Profit

Now ask the class, **What is Hassan's maximum possible profit? How can you be sure?** Roughly, students' reasoning should begin with

these ideas:

- The points that give a specific profit satisfy an equation that has a form similar to those above.

$$40p + 100w = \text{Profit}$$

- For any particular profit, these points will lie on a line that is parallel to the profit lines for Questions 2 through 4.
- Within this family of parallel lines, profit increases as the line moves up and to the right.

From here, students should intuitively understand that they want to "slide" one of these parallel lines up and to the right until it reaches the edge of feasible region. If the sketch is made carefully, students will see that among those lines in the family that intersect the feasible region, the most "extreme" line is the one through the point where the lines $p + w = 16$ and $5p + 15w = 180$ intersect. Therefore, the point where these two lines intersect represents the maximum profit. Because this reasoning is so visual, most students should be able to understand it, even if they didn't discover it on their own.

At least some students are likely to have found the coordinates of this point, (6, 10), either by guess-and-check or some other means. Ask**, How can you confirm that you have the right coordinates for the intersection point?** Have students verify that this point fits both equations. Then ask them to compute the profit for this combination of pictures, perhaps comparing it to any values they previously thought were optimal.

Because the profit lines are almost parallel to the line $5p + 15w = 180$, it may not be clear from students' graphs where the family of parallel lines leaves the feasible region. If so, ask, **If it's not clear from the graph, how can you decide which point maximizes profit?**

You might ask what other points seem likely. Students should realize from the graph that the point (0, 12) is also a reasonable candidate. **How can you be sure which is the right point if you don't trust your graph?** Students should recognize that they could compute the profit at the two points and compare. The point (6, 10), representing 6 pastels and 10 watercolors, gives a profit of $1,240, while the point (0, 12), representing 12 watercolors, gives a profit of only $1,200.

When Profit Lines Are Parallel to a Constraint Line

When the family of parallel lines is parallel to a side of the feasible region, all points on that side will give the same profit. You can have an interesting discussion about how one would make a decision in that case.

For example, Hassan might choose a point along that side based on what he likes to paint, because the profit is the same. In the unit problem, if one side of the feasible region were parallel to the family of parallel lines, the Woos might choose the point along that side that maximizes the number of plain cookies, because they think plain cookies are healthier.

In discussing the family of parallel lines, you might remind students that **slope** is a mathematical term related to "the amount of slant" a line has. Because parallel lines have "the same amount of slant," they have the same slope.

Finding the Point of Intersection

Students have realized that the point of maximum profit is the point where the lines $p + w = 16$ and $5p + 15w = 180$ meet. Ask them to share their methods for finding the coordinates of this intersection point.

These equations are simple enough that guess-and-check probably worked for finding the values of 6 and 10 for p and w, respectively. Encourage students to use guess-and-check as a reasonable first approach, but point out that in other situations, the method might not suffice, especially if the solution involves fractions.

Another good approach is to estimate from the graph. Because the coordinates are whole numbers in Hassan's problem, this method would give the exact values.

Finally, students can solve for the intersection symbolically, using methods they began developing in *The Overland Trail* with the help of the "mystery bags" model.

If students find the numbers by guess-and-check, urge them to check that the answer is a good approximation of the coordinates of the point of intersection on the graph. If they find the numbers by estimating the coordinates from the graph, they should check that the numbers do indeed satisfy both equations.

Whatever method they use, bring out that the values for p and w have two distinct but closely related properties:

- These numbers are the p- and w-coordinates of the point where the two lines intersect.
- These numbers are the values for p and w that satisfy both equations.

Key Questions

What do you notice about combinations that produce a given profit?

What happens as the profit increases?

What do you notice about the different sets of points?

Why do the points for a given profit lie on a straight line?

What do you notice about the set of profit lines?

What does it mean for lines to be parallel?

What is Hassan's maximum possible profit? How can you be sure?

How can you confirm that you have the right coordinates for the intersection point?

If it's not clear from the graph, how can you decide which point maximizes profit?

Curtis and Hassan Make Choices

Intent

This activity reinforces students' understanding of linear functions.

Mathematics

In this activity, students will find sets of inputs that produce specific outputs for two functions:

Cost = $2A + 3B$ (based on *Healthy Animals* and *Healthy Diets*)

Profit = $50p + 175w$ (based on *Picturing Pictures* and *Profitable Pictures*)

The conclusion they should draw in each case is that the points that produce a particular cost (or profit) are collinear.

Progression

Students work on this activity individually, likely for homework during the classwork on *Profitable Pictures.* They review their work as part of the discussion of that larger activity.

Approximate Time

20 minutes for activity (at home or in class)

15 minutes for discussion

Classroom Organization

Individuals

Doing the Activity

This activity requires little or no introduction.

Discussing and Debriefing the Activity

Review this activity as part of the discussion of *Profitable Pictures* in connection with parallel profit lines.